MATH 207(c)–Analysis IV

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1. Consider the following initial-boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \quad \text{for} \quad x \in (0,1),\, t > 0, \\ u(x,0) = \sin^2(2\pi x), \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Find the solution u expressed as a trigonometric series.

2. Consider the following initial-boundary value problem for the Klein–Gordon equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) - u(t,x) = 0 \quad \text{for} \quad x \in (0,1),\, t > 0, \\ u(x,0) = x(1-x), \quad \frac{\partial u}{\partial t}(x,0) = 0, \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Find the solution u expressed as a trigonometric series.

3. Consider the following initial-boundary value problem for the Schrödinger equation:

$$\begin{cases} i\frac{\partial u}{\partial t}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \quad \text{for} \quad x \in (0,1),\, t > 0, \\ u(x,0) = \sin^2(\pi x), \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Find the solution u expressed as a trigonometric series.

4. Let ψ be the solution to the following initial value problem for the biharmonic heat equation on the whole line:

$$\begin{cases} \frac{\partial \psi}{\partial t}(x,t) + \frac{\partial^4 \psi}{\partial x^4}(x,t) = 0 \quad \text{for} \quad x \in \mathbb{R},\, t > 0, \\ \psi(x,0) = \psi_0(x). \end{cases}$$

Show that the solution is given by the formula

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{t^{\frac{1}{4}}} \int_{-\infty}^{+\infty} G\left(\frac{x-y}{t^{\frac{1}{4}}}\right) \psi_0(y)\, dy,$$

where $G : \mathbb{R} \to \mathbb{R}$ is the function for which its Fourier transform is given by

$$\hat{G}(a) = e^{-a^4}.$$

(note that, a priori, the inverse Fourier transform of the above expression should be a function $G : \mathbb{R} \to \mathbb{C}$; show that G is indeed real valued, i.e. that $\overline{G(x)} = G(x)$ for any $x \in \mathbb{R}$.)

5. Let us consider the following *semi-infinite* initial-boundary value problem for the heat equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x) \quad \text{for} \quad x \in (0,+\infty),\, t>0, \\ u(x,0) = 0, \\ u(0,t) = 0, \end{cases}$$

where

$$f(x) = xe^{-\frac{x^2}{2}}.$$

By extending $u(x,t)$ and $f(x)$ as *odd* functions of $x \in \mathbb{R}$, solve the above problem by applying the Fourier transform in the x-variable. Verify that the solution $u(x,t)$ that you get in this way is indeed odd and that $u(x,t)$ satisfies the required boundary condition at $x=0$ (this should be automatically true for continuous odd functions).

6 (extra). Let $V : \mathbb{R} \to \mathbb{R}$ be a smooth function (which we will call the *potential*) and let $u : \mathbb{R} \times [0,+\infty) \to \mathbb{C}$ be a solution to the Schrödinger equation:

$$i\frac{\partial u}{\partial t}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) - V(x)u(x,t) = 0.$$

We will assume that, at any time $t \geqslant 0$, we have that $u(x,t)$, $\frac{\partial u}{\partial x}(x,t) \to 0$ as $x \to \pm\infty$.

(a) In the special case when $V(x) = 0$, find an expression for u if the initial data at $t=0$ is given by

$$u(x,0) = e^{i\lambda x - \frac{x^2}{2}}.$$

(b) In the general case (i.e. when V is not necessarily zero), show that the quantity

$$\int_{-\infty}^{+\infty} |u(x,t)|^2 \, dx$$

is constant in time (this motivates the interpretation of $|u(x,t)|^2$ as the probability density of the particle described by u). *Hint: Use the fact that* $|u|^2 = Re\{u \cdot \bar{u}\}$ *and show first that* $\partial_t |u|^2 = 2Re\{\partial tu \cdot \bar{u}\}$. *Then, use the equation to reexpress* $\partial_t u$ *and integrate by parts in* x *if necessary. Note that* $\partial(Re(f)) = Re(\partial f)$ *and* $\partial \bar{f} = \overline{\partial f}$.

(c) Show that the total energy of u, defined by

$$E[u](t) = \int_{-\infty}^{+\infty} \left(\left| \frac{\partial u}{\partial x}(x,t) \right|^2 + V(x) \, |u(x,t)|^2 \right) dx$$

is also constant in time. *Hint: Differentiate the above expression in t like before, integrate by parts with respect to the* ∂_x*-derivatives and use the equation to substitute for the* $\frac{\partial^2 u}{\partial x^2}$ *term.*